ELSEVIER Journal of Pure and Applied Algebra 127 (1998) 73–82

JOURNAL OF PURE AND APPLIED ALGEBRA

The dimension of the Hilbert scheme of Gorenstein codimension 3 subschemes

Jan O. Kleppe[a,1], Rosa M. Miró-Roig[b,*]

[a] *Oslo College, Faculty of Engineering, Cort Adelersgt 30, N-0254 Oslo, Norway*
[b] *Dept. Algebra y Geometria, Facultad de Matemáticas, Universidad de Barcelona, 08007 Barcelona, Spain*

Communicated by L. Robbiano; received 11 November 1996

Abstract

In this short note, we compute the dimension of the open subset of the Hilbert scheme, $Hilb_{p(t)}\mathbf{P}^n$, parametrizing AG closed subschemes $X \subset \mathbf{P}^n$ of codimension 3. © 1998 Elsevier Science B.V. All rights reserved.

1991 Math. Subj. Class: 14C05

0. Introduction

It is well known that Arithmetically Gorenstein (AG) closed subschemes $X \subset \mathbf{P}^n$ of codimension 3 are parametrized by smooth points of the Hilbert scheme $H = Hilb_{p(t)}\mathbf{P}^n$ (see [16]). The goal of this note is to compute the dimension of the local rings $\mathcal{O}_{H,[X]}$ at these points.

In 1960, Grothendieck [6] proved the existence of a projective $\mathbf{k}$-scheme, $H = Hilb_{p(t)}$ $\mathbf{P}^n$, parametrizing closed subschemes X of $\mathbf{P}^n$ with given Hilbert polynomial $p(t) \in \mathbf{Q}[t]$. Until now there are few general results about these schemes concerning connected components, dimension, smoothness, topological invariants, ... and they have only been studied for special polynomials $p(t) \in \mathbf{Q}[t]$ or in remarkable parts of $H = Hilb_{p(t)}\mathbf{P}^n$ (for instance, arithmetically Cohen–Macaulay closed subschemes of codimension 2, twisted cubics, ...).

In 1975, using the Hilbert–Burch structure theorem for homogeneous perfect ideals $I(X) \subset \mathbf{k}[X_0, \ldots, X_n]$ of codimension 2, Ellingsrud proved that Arithmetically

* Corresponding author. E-mail: miro@cerber.mat.ub.es. Partially supported by DGICYT PB94-0850.
[1] Partially supported by the Regional Board of Higher Education–Oslo and Akershus.

PII S0022-4049(96)00180-6

Cohen–Macaulay (ACM) closed subschemes $X \subset \mathbf{P}^n$ of codimension 2 are non-obstructed and he computed $dim\, \mathcal{O}_{H,[X]}$ in terms of the degrees of the syzygies of X [4]. Furthermore, we know exactly when two ACM closed subschemes $X \subset \mathbf{P}^n$ of codimension 2 belong to the same irreducible component of the Hilbert scheme and the polynomials which are Hilbert polynomials of some ACM closed subscheme $X \subset \mathbf{P}^n$ of codimension 2 [4,7]. Now we come to the AG codimension 3 case. By using Buchsbaum–Eisenbud's structure theorem for homogeneous ideals $I(X) \subset \mathbf{k}[X_0,\dots,X_n]$ of codimension 3 [1], we can prove that AG closed subschemes $X \subset \mathbf{P}^n$ of codimension 3 are non-obstructed (see [16] or [14, Proposition 3.12] because X is in the liaison class of a complete intersection [20]), we can characterize the Hilbert polynomials of AG closed subschemes $X \subset \mathbf{P}^n$ of codimension 3 (see [3, 17] and [19]) and decide when two AG closed subschemes $X \subset \mathbf{P}^n$ of codimension 3 belong to the same irreducible component of the Hilbert scheme (see [3]). We refer to [5] for other well-known results on ACM codimension 2 closed subschemes of projective space which have been shown to have striking analogs for codimension 3 AG closed subschemes.

However, it remains open the computation of $dim\, \mathcal{O}_{H,[X]}$. The problem is the following: Let $X \subset \mathbf{P}^n$ be an AG closed subscheme of codimension 3. What can be said about two $(2r+1)\times(2r+1)$ skew matrices M and N whose pfaffians generate $I(X)$?

In this paper, by using the structure theorem of Buchsbaum and Eisenbud, the fact that $I(X)$ is syzygetic and the explicit resolution of $\bigwedge^2 I(X)$ of Lebelt–Weyman, we compute the dimension of the open smooth subset of the Hilbert scheme, $Hilb_{p(t)}\mathbf{P}^n$, parametrizing AG closed subschemes $X \subset \mathbf{P}^n$ of codimension 3.

1. Notations and general facts

Throughout this paper we will work over an algebraically ground field $\mathbf{k}$ of characteristic $\neq 2$, $R = \mathbf{k}[X_0,\dots,X_n]$, $m = (X_0,\dots,X_n)$ and $\mathbf{P}^n = Proj(R)$. Given a closed subscheme $X \subset \mathbf{P}^n$, we denote by $\mathcal{I}_X$ (resp. $I = I(X) \subset R$) the ideal sheaf (the homogeneous ideal) of X, $N_X = \mathcal{H}om_{\mathcal{O}_{\mathbf{P}}}(\mathcal{I}_X, \mathcal{O}_X)$ the normal sheaf, $A = R/I(X)$ and $H^i(R,A,A)$ the corresponding i.algebra cohomology group of the graded morphism $R \to A$.

In the sequel, ${}_{\mu}Hom_R(M,-)$ denotes homomorphisms of graded R-modules of degree μ. If $\Gamma_m(M)$ is the group of sections of $M^{\sim}$ with support in $V(m) \subset Spec(R)$, i.e. $\Gamma_m(M)_\mu = Ker(M_\mu \to \Gamma(\mathbf{P}^n, M^{\sim}(\mu))$, we denote by $H^i_m(-)$ the right derived functors of $\Gamma_m(-)$.

For any closed subscheme $X \subset \mathbf{P}^n$ of codimension ≥ 3 one may use the well-known cotangent complex description of $H^2(R,A,A)$, as done in [13, Section 2.2] to prove:

$$Hom_R(I,I) \cong R,$$

$$H^1(R,A,A) \cong Hom_R(I,A) \cong Ext^1_R(I,I),$$

$$\Gamma_m(H^2(R,A,A)) \cong \Gamma_m(Ext^1_R(I,A)) \cong \Gamma_m(Ext^2_R(I,I))$$

and the isomorphisms preserve the grading (the proof is quite easy in the ACM case). Moreover, the isomorphisms involving Γ_m hold if we replace m by any graded prime ideal $\wp$ satisfying $depth A_{\wp} \geq 1$.

In the case X is an AG closed subscheme of $\mathbf{P}^n$ of codimension 3, there exists a minimal self-dual resolution of its homogeneous ideal of the following type [1]:

$$(*) \qquad 0 \to R(-f) \to \bigoplus_{i=1}^{r} R(-n_{2i}) \to \bigoplus_{i=1}^{r} R(-n_{1i}) \to I \to 0,$$

where $f = e+n+1$ and e by definition is the largest integer t such that $H^{n-3}(\mathcal{O}_X(t)) \neq 0$. The self-duality leads easily to $f - n_{2i} = n_{1i}$ provided we order the integers n_{ij} as

$$n_{11} \leq n_{12} \leq \cdots \leq n_{1r} \quad \text{and} \quad n_{21} \geq n_{22} \geq \cdots \geq n_{2r}.$$

Moreover, since $pd(I) = 2$, we get

$$Ext_R^2(I,I) \cong Ext_R^2(I,R) \otimes_R I \cong Ext_R^2(I,R) \otimes_A I/I^2,$$

i.e.

$$Ext_R^2(I,I) \cong I/I^2(f), \quad f = e + n + 1.$$

Similarly, for the corresponding sheaves, we have

$$N_X \cong \mathcal{E}xt^1(\mathcal{I}_X, \mathcal{I}_X),$$

$$\mathcal{I}_X/\mathcal{I}_X^2 \otimes_{\mathcal{O}_X} \omega_X(n+1) \cong \mathcal{E}xt^2(\mathcal{I}_X, \mathcal{I}_X), \quad \omega_X \cong \mathcal{O}_X(e).$$

2. Arithmetically Gorenstein subschemes of $\mathbf{P}^n$

We will begin this section with a result (Proposition 2.2) which rather explicitly describes the cohomology groups of $H^i(N_X(\mu))$ in case X is an AG closed subscheme of $\mathbf{P}^n$ of codimension 3. Later we compute the dimension of $H^0(N_X(\mu))$, thus determining $h^i(N_X(\mu)) = dim H^i(N_X(\mu))$ completely for any i and μ. As a special case we get the dimension of the Hilbert scheme at X. We will need:

Lemma 2.1. *Let $X \subset \mathbf{P}^n$ be an AG closed subscheme of codimension 3. Then, there exist exact (self-dual up to twist) sequences*

$$0 \to N_X \to \bigoplus \mathcal{O}_X(n_{1i}) \to \bigoplus \mathcal{O}_X(n_{2i}) \to \mathcal{I}_X/\mathcal{I}_X^2 \otimes \omega_X(n+1) \to 0$$

and

$$0 \to {}_{\mu}Ext_R^1(I,I) \to \bigoplus A_{n_{1i}+\mu} \to \bigoplus A_{n_{2i}+\mu} \to (I/I^2)_{e+n+1+\mu} \to 0$$

for any integer μ.

Proof. We consider the locally free resolution of $\mathcal{I}_X$

$$0 \to \mathcal{O}(-f) \to \bigoplus_{i=1}^{r} \mathcal{O}(-n_{2i}) \to \bigoplus_{i=1}^{r} \mathcal{O}(-n_{1i}) \to \mathcal{I}_X \to 0$$

obtained by sheafing the resolution $(*)$ of the homogeneous ideal $I = I(X)$ of X above. We set $K := Coker(R(-f) \to \bigoplus R(-n_{2i}))$, $f = e + n + 1$. Applying the functor $\mathcal{H}om_{\mathcal{O}_{\mathbf{P}^n}}(-, \mathcal{O}_X)$, we get

$$0 \to \mathcal{H}om(\mathcal{I}_X, \mathcal{O}_X) \to \bigoplus \mathcal{O}_X(n_{1i}) \to \mathcal{H}om(K^{\sim}, \mathcal{O}_X) \to \mathcal{E}xt^1(\mathcal{I}_X, \mathcal{O}_X) \to 0$$

and

$$0 \to \mathcal{H}om(K^{\sim}, \mathcal{O}_X) \to \bigoplus \mathcal{O}_X(n_{2i}) \to \mathcal{O}_X(f) \to \mathcal{E}xt^1(K^{\sim}, \mathcal{O}_X) \to 0$$

Since $\mathcal{E}xt^1(K^{\sim}, \mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{I}_X, \mathcal{O}_X) \cong \omega_X(n+1)$ and $\mathcal{O}_X(e) \cong \omega_X$, the map $\mathcal{O}_X(f) \to \mathcal{E}xt^1(K^{\sim}, \mathcal{O}_X)$ is an isomorphism. Hence, $\mathcal{H}om(K^{\sim}, \mathcal{O}_X) \cong \oplus \mathcal{O}_X(n_{2i})$. Furthermore, $\mathcal{E}xt^1(\mathcal{I}_X, \mathcal{O}_X) \cong \mathcal{E}xt^2(\mathcal{I}_X, \mathcal{I}_X) \cong \mathcal{I}_X/\mathcal{I}_X^2 \otimes \omega_X(n+1)$ and the first exact sequence easily follows. Now since A is a Gorenstein ring, we can repeat exactly the same proof above to the graded cones $R \to A = R/I$, replacing of course $\omega_X(n+1)$ by the canonical module $Ext^3_R(A,R) \cong A(f)$, and we get the second exact sequence. □

Proposition 2.2. *Let $X \subset \mathbf{P}^n$ be an AG closed subscheme of codimension 3. Then, $H^2(R,A,A) = 0 =$ and X is non-obstructed. Moreover, for any integer μ, we have $H^i(N_X(\mu)) = 0$ for $0 < i < n-3$ and the exact sequences*

$$0 \to H^{n-3}(N_X(\mu)) \to \bigoplus H^{n-3}(\mathcal{O}_X(n_{1i} + \mu)) \to \bigoplus H^{n-3}(\mathcal{O}_X(n_{2i} + \mu)) \to H^0(N_X(-\mu - n - 1))^v \to 0$$

and

$$0 \to H^0(N_X(\mu)) \to \bigoplus H^0(\mathcal{O}_X(n_{1i} + \mu)) \to \bigoplus H^0(\mathcal{O}_X(n_{2i} + \mu)) \to H^{n-3}(N_X(-\mu - n - 1))^v \to 0$$

Proof. We will deduce the vanishing of $H^2(R,A,A)$ from results of Huneke and Herzog. Indeed by [10, Corollary 1.13], $H^2(R,A,A)_{\wp} = 0$ for any ideal $\wp \in Proj(A)$ such that $dim A_{\wp} = 0$. The vanishing of $H^2(R,A,A)$ follows then from the Cohen–Macaulayness of I/I^2, cf. [8] or [2]. Indeed if $H^2(R,A,A) \neq 0$, then there exists a graded prime ideal $\wp \subset A$ such that $H^2(R,A,A)_{\wp} \neq 0$ and such that $H^2(R,A,A)_{\wp'} = 0$ for all graded prime ideal $\wp' \subsetneq \wp$. We get a contradiction (cf. Section 1) using $0 \neq H^2(R,A,A)_{\wp} = \Gamma_{\wp A_{\wp}}(H^2(R,A,A)_{\wp}) \cong \Gamma_{\wp A_{\wp}}(Ext^2(I,I)_{\wp}) = 0$; see also [11].

Now as pointed out in [16], the non-obstructedness of X follows from [11, Theorem 3.6 and Remark 3.7], because the deformation theories of $X \subset \mathbf{P}^n$ and $R \to A$ correspond uniquely in case $dim X \geq 1$; in the zero-dimensional case there is nothing to prove because $H^1(N_X) = 0$ and $H^2(R,A,A)^{\sim} = 0$.

Moreover, by the Cohen–Macaulayness of I/I^2, we get

$$H^i_m(I/I^2) \cong H^{i-1}_*(\mathcal{I}/\mathcal{I}^2) = 0 \quad \text{for } 2 \leq i \leq n-3$$

because $dim A = n-2$. Since

$$H^i(N_X(\mu)) \cong H^{n-3-i}(N_X^v(-\mu) \otimes \omega_X)^v \cong H^{n-3-i}(\mathcal{I}_X/\mathcal{I}_X^2(e-\mu))^v$$

we get $H^i(N_X(\mu)) = 0$ for $0 < i < n-3$.

It remains to prove the exact sequences. If $\dim X = 0$, i.e. $n = 3$, we conclude by taking global sections of the first exact sequence of Lemma 2.1 and using duality. Finally if $dim X \geq 1$, the Cohen–Macaulayness of I/I^2 implies $H^i_m(I/I^2) = 0$ for $i = 0, 1$, i.e.

$$\begin{aligned}(I/I^2)_{e+n+1+\mu} &\cong H^0(\mathscr{I}_X/\mathscr{I}_X^2(e+n+1+\mu)) \\ &\cong H^{n-3}((\mathscr{I}_X/\mathscr{I}_X^2)^\vee(-n-1-\mu))^\vee.\end{aligned}$$

Hence, we get one of the exact sequences from Lemma 2.1. Now dualizing this exact sequence, we get the other exact sequence because $\oplus H^0(\mathscr{O}_X(n_{1i}+\mu))^\vee \cong \oplus H^{n-3}(\omega_X(-n_{1i}-\mu)) \cong \oplus H^{n-3}(\mathscr{O}_X(e-n_{1i}-\mu))$ and $e+n+1-n_{1i} = n_{2i}$. □

Remark 2.3. If $X \subset \mathbf{P}^n$ is a closed, locally Gorenstein and equidimensional subscheme of codimension 3, one may by the proof above see that the sheaf $H^2(R,A,A)^\sim$ vanishes (see also [11, Corollary 4.11]).

Proposition 2.4. *Let $X \subset \mathbf{P}^n$ be an AG closed subscheme of codimension* 3. *Then, there is an exact sequence*

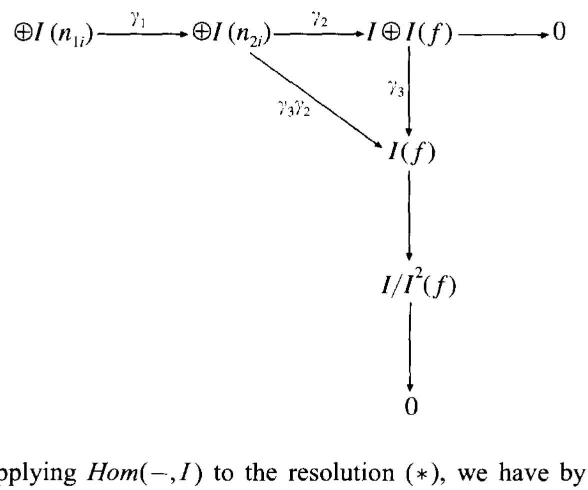
$$0 \to Ext^1_R(I,I) \to (I \otimes_R I)(f) \to I(f) \to (I/I^2)(f) \cong Ext^2_R(I,I) \to 0,$$

where $f = e+n+1$. Moreover, $Ext^1_R(I,I) \cong (\bigwedge^2 I)(f)$.

Proof. Twisting the exact sequence $(*)$ of Section 1 by f, we get the exact sequence

$$0 \to R \to \bigoplus_{i=1}^{r} R(n_{1i}) \to \bigoplus_{i=1}^{r} R(n_{2i}) \to I(f) \to 0$$

which we tensor with I and we obtain



Applying $Hom(-,I)$ to the resolution $(*)$, we have by definiton

$$Ext^1(I,I) \cong ker(\gamma_3\gamma_2)/\mathrm{im}(\gamma_1).$$

Thus, $Ext^1(I,I) \cong ker(\gamma_3)$ and the first exact sequence is proved. Finally, using the fact that I is a syzygetic ideal [18], one knows that the sequence

$$0 \to \bigwedge^2 I \to I \otimes I \to I^2 \to 0$$

is exact because 2 is invertible in R, and the conclusion of (2.4) is proved. □

The following useful remark will give us a finite free resolution (of length 3) of $\bigwedge^2 I$:

Remark 2.5. By Weyman [21], the sequence (involving graded pieces of the divided power algebra):

$$0 \to D_1F_1 \otimes F_2 \to (F_0 \otimes F_2) \oplus D_2F_1 \to F_0 \otimes D_1F_1 \to \bigwedge^2 F_0 \to \bigwedge^2 I \to 0$$

is exact, provided

$$0 \to F_2 = R(-f) \to F_1 \to F_0 \to I \to 0$$

is the exact sequence $(*)$ of Section 1.

Finally, combining Proposition 2.4 and Remark 2.5, we will compute for any AG closed subscheme $X \subset \mathbf{P}^n$ of codimension 3, the dimension of the Hilbert scheme at X, $dim\, Hilb_{[X]}\mathbf{P}^n$, in terms of the degrees of the syzygies of X. In Theorem 2.6 we have used the convention $\binom{b+n}{n} = 0$ for $b < 0$, and the numbers n_{ij} are ordered as mentioned in Section 1.

Theorem 2.6. *Let $X \subset \mathbf{P}^n, n > 3$, be an AG closed subscheme of codimension 3 whose homogeneous ideal $I = I(X)$ has a minimal free resolution of the following type:*

$$(*) \qquad 0 \to F_2 = R(-f) \to F_1 = \bigoplus_{i=1}^{r} R(-n_{2i}) \to F_0 = \bigoplus_{i=1}^{r} R(-n_{1i}) \to I \to 0.$$

Then, for any integer μ, we have

(1) $h^0(N_X(\mu)) = \sum_{i=1}^{r} h^0(\mathcal{O}_X(n_{1i} + \mu)) + dim(\bigwedge^2 F_0)_{f+\mu} - dim(\bigwedge^2 F_1)_{f+\mu} - dim(F_1)_{f+\mu} + dim(F_0)_{\mu}$,

(2) $h^i(N_X(\mu)) = 0$ *for* $0 < i < n-3$, *and*

(3) $h^{n-3}(N_X(\mu)) = \sum_{i=1}^{r} h^{n-3}(\mathcal{O}_X(n_{1i} + \mu)) + dim(\bigwedge^2 F_0)_{e-\mu} - dim(\bigwedge^2 F_1)_{e-\mu} - dim(F_1)_{e-\mu} + dim(F_0)_{-n-1-\mu}$.

In particular,

$$dim\, Hilb_{[X]}\mathbf{P}^n = h^0(N_X) = \sum_{i=1}^{r} h^0(\mathcal{O}_X(n_{1i})) + \sum_{1 \le i < j \le r} \binom{-n_{1_i} + n_{2_j} + n}{n} - \sum_{1 \le i < j \le r} \binom{n_{1_i} - n_{2_j} + n}{n} - \sum_{i=1}^{r} \binom{n_{1i} + n}{n}.$$

Remark. If $n=3$, the final dimension formula is the dimension of locally closed subschemes of $\mathbf{P}^3$ consisting of graded Gorenstein R-algebra quotients with fixed Hilbert function provided we replace $h^0(\mathcal{O}_X(n_{1i}))$ by $dim A_{n_{1i}}$.

Proof. Using Remark 2.5, $D_2F_1=S_2F_1$ (symmetric algebra square) and $D_1F_j=F_j$, we get the exact sequence

$$0\to F_1\otimes F_2\to (F_0\otimes F_2)\oplus S_2F_1\to F_0\otimes F_1\to \bigwedge^2 F_0\to \bigwedge^2 I\to 0.$$

Since $F_i\otimes F_2(f)=F_i$, it follows from Proposition 2.4 that

$$dim({}_{\mu}Ext^1(I,I)) = dim\left(\bigwedge^2 F_0\right)_{f+\mu} - dim(F_0\otimes F_1)_{f+\mu} + dim(S_2F_1)_{f+\mu} + dim(F_0)_{\mu} - dim(F_1)_{\mu}.$$

Since $F_1(f)=\oplus R(f-n_{2i})=\oplus R(n_{1i})$, tensoring the exact sequence $(*)$ with $F_1(f)$, we get

$$0\to F_2\otimes F_1(f)\to F_1\otimes F_1(f)\to F_0\otimes F_1(f)\to \oplus I(n_{1i})\to 0$$

which together with the isomorphism $\bigwedge^2 F_1\oplus S_2F_1\cong F_1\otimes F_1$ gives

$$\begin{aligned} dim_{\mu}\, Ext^1(I,I) &= dim\left(\bigwedge^2 F_0\right)_{f+\mu} - \sum h^0\mathcal{I}_X(n_{1i}+\mu) - dim(F_1\otimes F_1)_{f+\mu} \\ &\quad + dim(S_2F_1)_{f+\mu} + dim(F_0)_{\mu} \\ &= dim\left(\bigwedge^2 F_0\right)_{f+\mu} - \sum h^0\mathcal{I}_X(n_{1i}+\mu) - dim\left(\bigwedge^2 F_1\right)_{f+\mu} \\ &\quad + dim(F_0)_{\mu} \end{aligned}$$

and we easily get the formula of $h^0(N_X(\mu))=dim({}_{\mu}Ext^1(I,I))$ as stated. In particular, since X is non-obstructed, it follows that

$$\begin{aligned} dim\, Hilb_{[X]}\mathbf{P}^n &= h^0(N_X) \\ &= \sum_{i=1}^{r} h^0(\mathcal{O}_X(n_{1i})) + dim\left(\bigwedge^2 F_0\right)_f - dim\left(\bigwedge^2 F_1\right)_f - dim(F_1)_f. \end{aligned}$$

But

$$\left(\bigwedge^2 F_0\right) = \bigoplus_{1\le i<j\le r} R(-n_{1_i} - n_{1_j}),$$

$$\left(\bigwedge^2 F_0\right)_f = \bigoplus_{1\le i<j\le r} R_{-n_{1_i}-n_{1_j}+f} = \bigoplus_{1\le i<j\le r} R_{-n_{1_i}+n_{2_j}},$$

$$\left(\bigwedge^2 F_1\right)_f = \bigoplus_{1\le i<j\le r} R_{-n_{2_i}+n_{1_j}}.$$

Hence,

$$dim\left(\bigwedge^2 F_0\right)_f - dim\left(\bigwedge^2 F_1\right)_f - dim(F_1)_f$$

$$= \sum_{1\le i<j\le r}\binom{-n_{1_i}+n_{2_j}+n}{n} - \sum_{1\le i<j\le r}\binom{n_{1_i}-n_{2_j}+n}{n} - \sum_{i=1}^{r}\binom{n_{1_i}+n}{n}.$$

Finally, to show the formulas of $h^i(N_X(\mu))$, $i>0$, we use Proposition 2.2. We get the vanishing of $h^i(N_X(\mu))$ in the case $0<i<n-3$ and moreover

$$h^0(N_X(-\mu-n-1)) - \sum h^{n-3}(\mathcal{O}_X(n_{2i}+\mu))$$

$$= h^{n-3}(N_X(\mu)) - \sum h^{n-3}(\mathcal{O}_X(n_{1i}+\mu)).$$

Since $h^{n-3}(\mathcal{O}_X(n_{2i}+\mu)) = h^0(\mathcal{O}_X(e-n_{2i}-\mu)) = h^0(\mathcal{O}_X(n_{1i}-n-1-\mu))$, we can easily conclude using the proven expression of $h^0(N_X(\mu))$. □

Remark 2.7. For a global complete intersection $X \subset \mathbf{P}^n$ of type (n_1, n_2, n_3) we deduce from Theorem 2.6 the well-known formula

$$h^0(N_X) = \sum_{i=1}^{3} h^0(\mathcal{O}_X(n_i)) \quad \text{and} \quad h^{n-3}(N_X) = \sum_{i=1}^{3} h^{n-3}(\mathcal{O}_X(n_i)).$$

Remark 2.8. Now it is easy to find an expression of $h^0(N_X)$ or, equivalently, of $dim\,Hilb_{[X]}\mathbf{P}^n$ which does not involve $\sum h^0(\mathcal{O}_X(n_{1i}))$. For instance, using the first exact sequence in the proof above, together with the expression of $\bigwedge^2 F_0$ (and corresponding expressions of S_2F_1 and $F_0 \otimes F_1$) appearing later in the proof, we get

$$h^0(N_X) = \sum_{1\le i<j\le r}\binom{-n_{1_i}+n_{2_j}+n}{n} - \sum_{1\le i,j\le r}\binom{-n_{1_i}+n_{1_j}+n}{n}$$

$$+ \sum_{1\le i\le j\le r}\binom{n_{1i}-n_{2_j}+n}{n}.$$

Example 2.9. Consider the AG curves $X \subset \mathbf{P}^4$ whose homogeneous ideal has a resolution of the following type (see [9, Theorem 1.2] for the existence of such smooth curves):

$$0 \to R(-10) \to R(-6)^5 \oplus R(-5)^2 \to R(-5)^2 \oplus R(-4)^5 \to I \to 0.$$

We easily get $h^0(\mathcal{O}_X(\mu)) = h^1(\mathcal{O}_X(5-\mu)) = \binom{\mu+4}{4}$ for $0 \le \mu \le 3$. Hence $\chi(\mathcal{O}_X(2)) = -20$ and $\chi(\mathcal{O}_X(3)) = 20$. By Riemann–Roch's theorem

$$d = deg(X) = 40, \qquad g = gen(X) = 101.$$

Now to compute $dim\, Hilb_{[X]}\mathbf{P}^4$, we use Theorem 2.6. Inserting $h^0(\mathcal{I}_X(n_{1i})) = h^0(\mathcal{O}_{\mathbf{P}}(n_{1i})) - h^0(\mathcal{O}_X(n_{1i}))$, we get

$$h^0(N_X) = \sum_{1 \le i < j \le 7} \left(\binom{n_{2j} - n_{1i} + 4}{4} - \binom{n_{1i} - n_{2j} + 4}{4} \right) - \sum_{i=1}^{7} h^0(\mathcal{I}_X(n_{1i})) = 125$$

recalling $n_{11} \le n_{12} \le \cdots \le n_{17}$ and $n_{21} \ge n_{22} \ge \cdots \ge n_{27}$. The formula for $h^1(N_X)$ of Theorem 2.6 leads to

$$h^1(N_X) = 5h^1(\mathcal{O}_X(4)) + 2h^1(\mathcal{O}_X(5)) - 2 = 25$$

which again implies $h^0(N_X) = 125$ because $\chi(N_X) = 5d + 1 - g = 100$.

Remark 2.10. Using Theorem 2.6, one may see that

$$h^{n-3}(N_X) = \sum_{i=1}^{r} \left(\binom{n_{2i} - 1}{n} - \binom{n_{1i} - 1}{n} \right) \quad \text{provided } e < 2\min(n_{1i}).$$

For the example above, this gives immediately $h^1(N_X) = 25$.

For proving the formula, we will first deduce a vanishing result of $h^0(N_X(-n-1))$ using Theorem 2.6. Indeed this last group vanishes provided

(1) $n_{2j} - n_{1i} < n + 1$ for any i, j,

(2) $-n_{2j} + n_{1i} < n + 1$ for any i, j, and

(3) $H^0(\mathcal{I}_X(n_{1i} - n - 1)) = 0$ for any i

because (3) is equivalent to $\sum h^0(\mathcal{O}_X(n_{1i} - n - 1)) = dim(F_1)_{f-n-1}$. Now note that (1) implies $H^0(\mathcal{I}_X(n_{2j} - n - 1)) = 0$ for any j; hence (1) implies (3). Moreover, (1) is equivalent to $\max(n_{2j}) < n + 1 + \min(n_{1i})$, and using $\max(n_{2j}) = f - \min(n_{1i})$, we see that (1) means exactly $e < 2\min(n_{1i})$. Since (2) is similarly equivalent to $e < 2\min(n_{2i})$, we see that (1) implies (2). Hence if $e < 2\min(n_{1i})$, the group $H^0(N_X(-n-1))$ vanishes. Now we conclude by Proposition 2.2,

$$\begin{aligned} h^{n-3}(N_X) &= \sum_{i=1}^{r} h^0(\mathcal{O}_X(n_{2i} - n - 1)) - \sum_{i=1}^{r} h^0(\mathcal{O}_X(n_{1i} - n - 1)) \\ &= \sum_{i=1}^{r} \left(\binom{n_{2i} - 1}{n} - \binom{n_{1i} - 1}{n} \right), \end{aligned}$$

where we have used (3) and $H^0(\mathscr{I}_X(n_{2j}-n-1))=0$ for any j, to see the equality to the right-hand side.

2.11. In a forthcoming paper, we will come to the ACM codimension 3 case. We will give sufficient conditions for assuring the non-obstructedness of an ACM curve in $\mathbf{P}^4$ and in some cases we will compute the dimension of the Hilbert scheme. Furthermore, we will give examples of obstructed ACM curves in $\mathbf{P}^4$ and we will describe infinitely many different liaison classes containing ACM curves [15].

Acknowledgements

This paper was written in the context of the group "Space Curves" of Europroj. The second author thanks the Department of Mathematics of the University of Oslo for its hospitality during the preparation of part of this work.

References

[1] D. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math. 99 (1977) 447–485.
[2] R.O. Buchweitz, Contributions à la théorie des singularités 3, Ph.D. Thesis, Univ. Paris VII, 1981.
[3] S. Diesel, Irreducibility and dimension theorems for families of height 3 Gorenstein Algebras, Pacific J., to appear.
[4] G. Ellingsrud, Sur le schéme d'Hilbert des variétès de codimension 2 à cône de Cohen–Macaulay, Ann. Sc. Ec. Norm. Sup. 2 (1975) 423–432.
[5] A. Geramita and J. Migliore, Reduced Gorenstein codimension 3 subschemes of projective spaces, preprint, 1995.
[6] A. Grothendieck, Les schemas de Hilbert, Sem. Bourbaki Exp. 22 (1960).
[7] L. Gruson and C. Peskine, Genre des courbes de l'espace projectif, Lecture Notes in Math., Vol. 687 (Springer, Berlin, 1978) 31–59.
[8] J. Herzog, Ein Cohen–Macaulay Kriterium mit Anwendungen auf den Konormalen Modul und den Differential modul, Math. Z. 1963 (1978) 149–162.
[9] J. Herzog, N.V. Trung and G. Valla, On hyperplane sections of reduced irreducible varieties of low codimension, J. Math. Kyoto Univ. 34 (1994) 47–72.
[10] C. Huneke, Invariants of liaison, in: Algebraic Geometry, Proc. Ann Arbor, 1981, Lecture Notes in Math., Vol. 1008 (Springer, Berlin, 1983) 65–74.
[11] C. Huneke, Numerical invariants of liaison classes, Inv. Math. 75 (1984) 301–325.
[12] J.O. Kleppe, Deformations of graded algebras, Math. Scand. 45 (1979) 205–231.
[13] J.O. Kleppe, The Hilbert flag scheme, its properties and its connection with the Hilbert scheme, Ph.D. thesis, 1981.
[14] J.O. Kleppe, Liaison of families of subschemes of $\mathbf{P}^n$, Proc. Trento, 1988, Lecture Notes in Math., Vol. 1389 (Springer, Berlin, 1989) 128–173.
[15] J.O. Kleppe and R.M. Miró-Roig, in preparation.
[16] R.M. Miró-Roig, Non-obstructedness of Gorenstein Subschemes of codimension 3 in $\mathbf{P}^n$, Beitrage Alg. Geom. 33 (1992) 131–138.
[17] E. De Negri and G. Valla, The h-vector of a Gorenstein codimension 3 domain, preprint, 1994.
[18] A. Simis and W. Vasconcelos, The syzygies of the conormal bundle, Amer. J. Math. 103 (1981) 203–224.
[19] R.P. Stanley, Hilbert functions of graded algebras, Adv. Math. 28 (1978) 57–83.
[20] S. Watanabe, A note on Gorenstein rings of embedding codimension 3, Nagoya Math. J. 50 (1973) 227–232.
[21] Weyman, Resolutions of the exterior and symmetric power of a module, J. Algebra 58 (1979) 333–341.